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Exact analysis of localized modes in two-dimensional bi-periodic mass–spring systems with a single disorder

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Abstract

The frequency equation and localized modes in two-dimensional bi-periodic mass–spring systems with one disordered subsystem are exactly analyzed by means of double U-transformation.

At first a plane distributed mass–spring system with $2n_1 \times 2n_2$ subsystems and cyclic periodicity in x- and y-directions is considered. Then by adopting a limiting process with n_1, n_2 approaching to infinity, the limiting solution is applicable for the plane distributed bi-periodic mass–spring systems with boundary at infinity.

The explicit frequency equation and localized modes are derived. Some specific systems are taken as examples to demonstrate how to apply the formulas and equations obtained in the present paper in order to find the localized modes and corresponding frequencies.

1. Introduction

Analyses of bi-periodic systems have been studied by using various methods, including transfer matrix method [1], wave approach [2–4], standard stiffness and transmission methods [5] and U-transformation method [6–10]. Vibration analyses of disordered periodic systems have been investigated by Bansal [11] and Mead et al. [12–14] using receptance method.

Localization phenomenon was first predicted by Anderson [15] in the field of solid-state physics. In structural dynamics, Hodges [16] was the earliest to study localized modes in one-dimensional periodic

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structures. There is a great volume of literature on localization. A detailed discussion of that literature is contained in the special issue of Chaos, Solitons and Fractals on localization problems [17].

Mode localization phenomenon in infinite periodic mass–spring systems having one linear or nonlinear disorder was investigated by Cai et al. [18–20] using the U-transformation method. Recently the U-transformation method was extended to analyze the localized modes in infinite bi-periodic mass–spring systems with a single disorder [21]. In order to derive the frequency equation for localized modes, the U-transformation must be used twice. In the present study, two-dimensional bi-periodic mass–spring systems having one disorder are considered. In order to uncouple the governing equation of the bi-periodic systems, the double U-transformation needs to be used twice [22].

2. Governing equation

Consider a two-dimensional infinite bi-periodic mass–spring system with a single disorder as shown in Fig. 1. This system consists of two different kinds of subsystems, say M- and M'-subsystems, where only one subsystem departs from the regularity in both stiffness and mass. M, M' and $M' + M_d$ denote the masses, and K, K' and $K' + K_d$ denote the stiffness for M-, M'- and disordered subsystems, respectively. In Fig. 1, two sets of pretensioned straight strings with fixed ends at infinity act as the coupling springs between two adjacent subsystems in x- and y-directions. The stiffness of coupling spring in x- and y-directions can be expressed as $K_1 = T_1/a$ and $K_2 = T_2/b$, respectively, where T_1, T_2 denote the pretentions of the strings in x- and y-directions and a, b denote the spacing of y- and x-strings, respectively.

The localized modes in an infinite periodic system are negligibly affected by the conditions at infinity. Consequently, the system under consideration may be regarded as cyclic bi-periodic as shown in Fig. 2. At first, the two-dimensional cyclic bi-periodic system with $2n_1 \times 2n_2$ subsystems is considered. In Fig. 2, a pair of integers (j, k) denote a subsystem in which j and k in the round brackets denote the ordinal numbers of the subsystem along x- and y-directions, respectively. The identical relations $(0, k) \equiv (2n_1, k)$, $k = 0, 1, 2, \ldots, 2n_2$ and $(j, 0) \equiv (j, 2n_2)$, $j = 0, 1, 2, \ldots, 2n_1$ represent the cyclic periodicity in x- and y-directions, respectively. Then by adopting a limiting process with n_1, n_2 approaching to infinity, the governing equation and its solution will be applicable for the original system shown in Fig. 1.

Applying Newton's second law to every subsystem, the natural vibration equations can be expressed as

$$(K + 2K_1 + 2K_2 - M\omega^2)w_{(j,k)} - K_1(w_{(j+1,k)} + w_{(j-1,k)}) - K_2(w_{(j,k+1)} + w_{(j,k-1)}) = F_{(j,k)}, \quad j = 1, 2, \ldots, 2n_1, \quad k = 1, 2, \ldots, 2n_2 \tag{1a}$$

and

$$F_{(j,k)} = \begin{cases} (\Delta M\omega^2 - \Delta K)w_{(j,k)}, & (j,k) = (j_1p_1, k_1p_2) \text{ and } (j,k) \neq (n_1, n_2), \\ [(\Delta M + M_d)\omega^2 - (\Delta K + K_d)]w_{(j,k)}, & (j,k) = (n_1, n_2), \\ 0, & (j,k) \neq (j_1p_1, k_1p_2), \end{cases}$$
$$n_1 \equiv m_1p_1, \quad n_2 \equiv m_2p_2; \quad j_1 = 1, 2, \ldots, 2m_1, \quad k_1 = 1, 2, \ldots, 2m_2, \tag{1b}$$

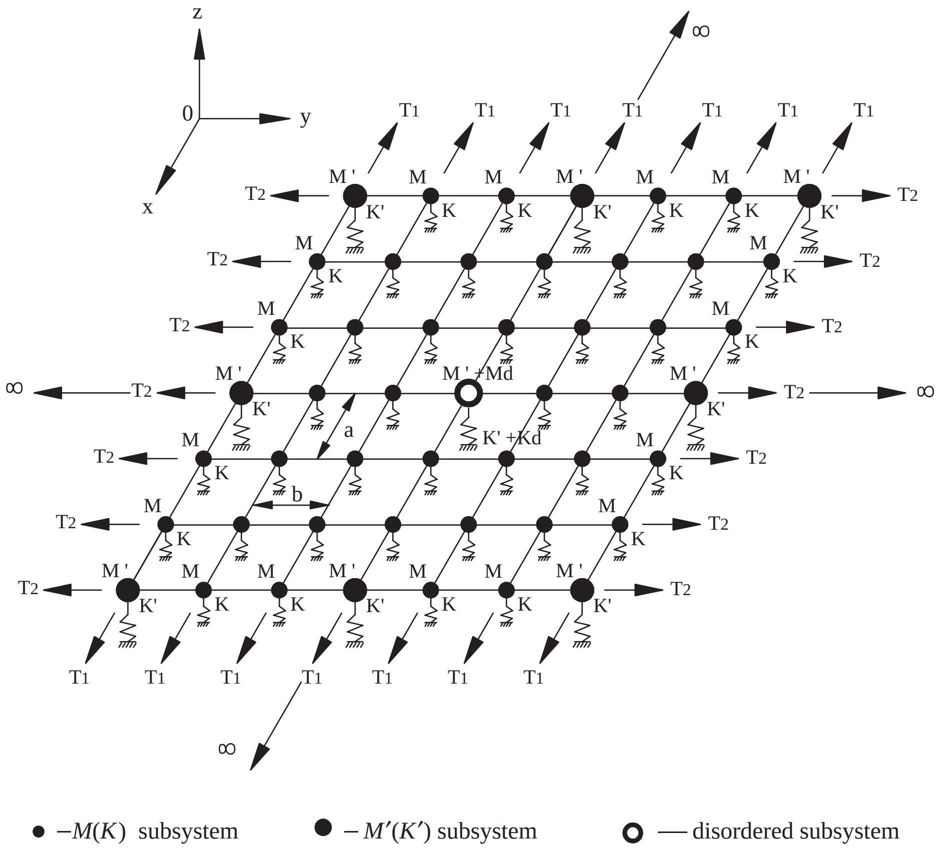


Fig. 1. Two-dimensional bi-periodic mass–spring system with one disorder.

where $w_{(j,k)}$ denotes the amplitude of displacement (in z-direction) for the (j, k) subsystem; ω represents the natural frequency; M_d, K_d denote the magnitudes of disorder for mass and stiffness and $\Delta M = M' - M$, $\Delta K = K' - K; (j_1p_1, k_1p_2)$, $j_1 = 1, 2, \ldots, 2m_1$, $k_1 = 1, 2, \ldots, 2m_2$ represents the M'-subsystems, and (m_1p_1, m_2p_2) denotes the disordered one. Without loss of generality, it is assumed that the disordered subsystem is located at the center of the considered system as shown in Fig. 2. In Eq. (1a), it should be noted that $w_{(2n_1+1,k)} \equiv w_{(1,k)}$, $w_{(0,k)} \equiv w_{(2n_1,k)}$ and $w_{(j,2n_2+1)} \equiv w_{(j,1)}$, $w_{(j,0)} \equiv w_{(j,2n_2)}$ due to the cyclic periodicity in x- and y-directions.

3. The first application of the double U-transformation

The left-hand sides of Eq. (1a) possess cyclic periodicity for two subscripts in the round brackets where the two subscripts represent two ordinal numbers of the subsystem and $F_{(j,k)}$ on the right-hand sides of Eq. (1a) act as the loads. In order to uncouple the left-hand sides of the simultaneous equations (1a), one can now apply the double U-transformation [23] to Eq. (1a).

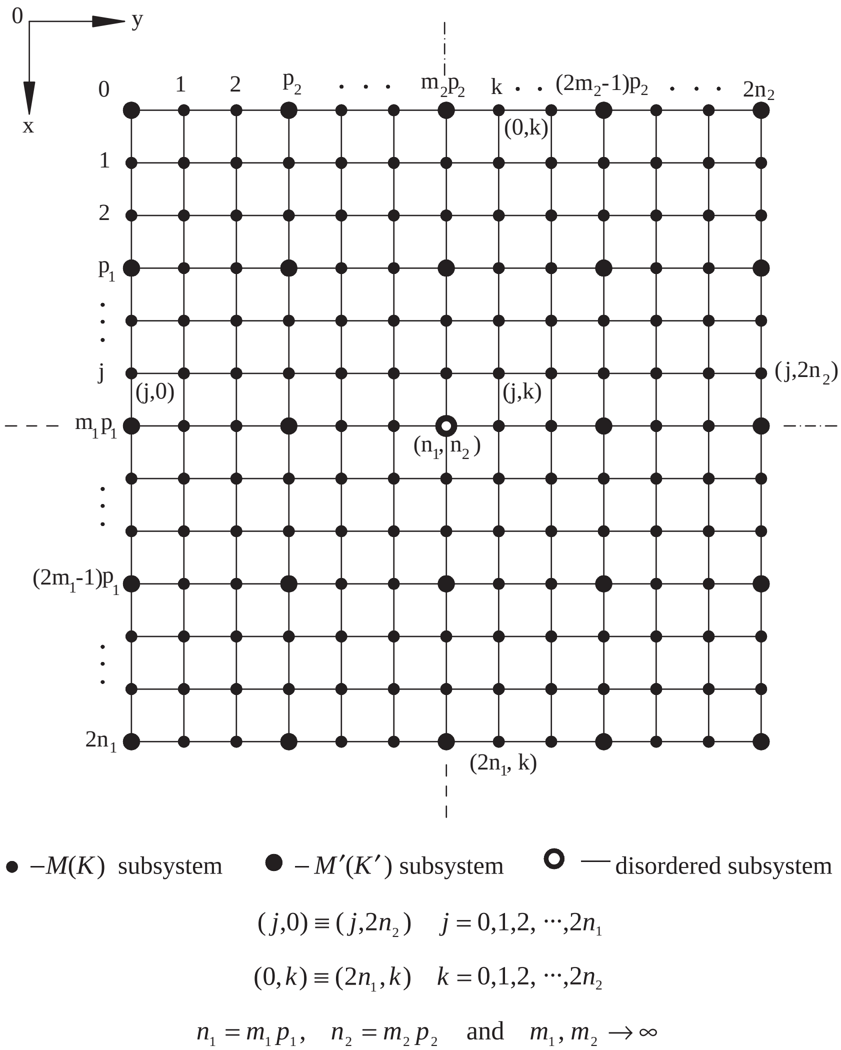


Fig. 2. Equivalent system with $2n_1 \times 2n_2$ subsystems and cyclic bi-periodicity in x- and y-directions.

Let

$$q_{(r,s)} = \frac{1}{\sqrt{2n_1}\sqrt{2n_2}} \sum_{j=1}^{2n_1} \sum_{k=1}^{2n_2} \mathrm{e}^{-\mathrm{i}(j-1)r\psi_1} \mathrm{e}^{-\mathrm{i}(k-1)s\psi_2} w_{(j,k)}, \quad r = 1, 2, \ldots, 2n_1, \quad s = 1, 2, \ldots, 2n_2 \tag{2a}$$

and its inverse transformation is

$$w_{(j,k)} = \frac{1}{\sqrt{2n_1}\sqrt{2n_2}} \sum_{r=1}^{2n_1} \sum_{s=1}^{2n_2} \mathrm{e}^{\mathrm{i}(j-1)r\psi_1} \mathrm{e}^{\mathrm{i}(k-1)s\psi_2} q_{(r,s)}, \quad j = 1, 2, \ldots, 2n_1, \quad k = 1, 2, \ldots, 2n_2, \tag{2b}$$

where $\mathrm{i} = \sqrt{-1}$, $\psi_1 = \pi/n_1$, $\psi_2 = \pi/n_2$ and ψ_1, ψ_2 denote the periods of the cyclic periodic system for M-subsystems in x- and y-directions, respectively.

The right-hand side of Eq. (2b) can be regarded as the series of rotating modes [24] for the ordered cyclic periodic system (i.e., $\Delta M = 0$, $\Delta K = 0$, $M_d = 0$, $K_d = 0$) and $r\psi_1$, $s\psi_2$ denote the phase differences between the two adjacent subsystems in x- and y-directions, respectively. $q_{(r,s)}$ ($r = 1, 2, \ldots, 2n_1,\ s = 1, 2, \ldots, 2n_2$) is a set of generalized displacements.

The natural vibration equations (1a) can be expressed in terms of the generalized displacements $q_{(r,s)}$ as

$$(K + 2K_1 + 2K_2 - M\omega^2 - 2K_1 \cos r\psi_1 - 2K_2 \cos s\psi_2) q_{(r,s)} = f_{(r,s)}, \quad r = 1, 2, \ldots, 2n_1, \quad s = 1, 2, \ldots, 2n_2, \tag{3a}$$

where

$$f_{(r,s)} = \frac{\Delta M\omega^2 - \Delta K}{\sqrt{2n_1}\sqrt{2n_2}} \sum_{j_1=1}^{2m_1} \sum_{k_1=1}^{2m_2} \mathrm{e}^{-\mathrm{i}(j_1 p_1 - 1) r\psi_1} \mathrm{e}^{-\mathrm{i}(k_1 p_2 - 1) s\psi_2} w_{(j_1 p_1, k_1 p_2)} + \frac{M_d\omega^2 - K_d}{\sqrt{2n_1}\sqrt{2n_2}} \mathrm{e}^{-\mathrm{i}(n_1 - 1) r\psi_1} \mathrm{e}^{-\mathrm{i}(n_2 - 1) s\psi_2} w_{(n_1, n_2)}. \tag{3b}$$

Introducing $q_{(r,s)}$ obtained from Eqs. (3a) and (3b) into Eq. (2b) results in

$$w_{(j,k)} = (\Delta M\omega^2 - \Delta K) \sum_{j_1=1}^{2m_1} \sum_{k_1=1}^{2m_2} \beta^0_{(j,k)(j_1 p_1, k_1 p_2)} w_{(j_1 p_1, k_1 p_2)} + (M_d\omega^2 - K_d) \beta^0_{(j,k)(n_1, n_2)} w_{(n_1, n_2)}, \quad j = 1, 2, \ldots, 2n_1, \quad k = 1, 2, \ldots, 2n_2 \tag{4}$$

and

$$\beta^0_{(j,k)(u,v)} = \frac{1}{4n_1 n_2} \sum_{r=1}^{2n_1} \sum_{s=1}^{2n_2} \frac{\mathrm{e}^{\mathrm{i}(j-u) r\psi_1} \mathrm{e}^{\mathrm{i}(k-v) s\psi_2}}{K + 2K_1 + 2K_2 - M\omega^2 - 2K_1 \cos r\psi_1 - 2K_2 \cos s\psi_2}, \tag{5}$$

which denotes the harmonic influence coefficient (i.e., receptance) for the single perfectly periodic system with the parameters ΔK, ΔM, K_d and M_d vanishing.

Let

$$W_{(j,k)} \equiv w_{(jp_1, kp_2)}, \quad j = 1, 2, \ldots, 2m_1, \quad k = 1, 2, \ldots, 2m_2. \tag{6}$$

Here $W_{(j,k)}$ denotes the displacement for $(j,k)M'$-subsystem, i.e., the jth (kth) M'-subsystem in x- (y-) direction.

From Eq. (4), we can obtain the simultaneous equations with unknowns $W_{(j,k)}$ as

$$W_{(j,k)} - (\Delta M\omega^2 - \Delta K) \sum_{u=1}^{2m_1} \sum_{v=1}^{2m_2} \beta_{(j,k)(u,v)} W_{(u,v)} = W^*_{(j,k)}, \quad j = 1, 2, \ldots, 2m_1, \quad k = 1, 2, \ldots, 2m_2 \tag{7a}$$

and

$$W^*_{(j,k)} = (M_d\omega^2 - K_d) \beta_{(j,k)(m_1, m_2)} W_{(m_1, m_2)}, \tag{7b}$$

$$\beta_{(j,k)(u,v)} \equiv \beta^0_{(jp_1,kp_2)(up_1,vp_2)} = \frac{1}{4n_1n_2}\sum_{r=1}^{2n_1}\sum_{s=1}^{2n_2}\frac{e^{i(j-u)r\varphi_1}e^{i(k-v)s\varphi_2}}{K+2K_1+2K_2-M\omega^2-2K_1\cos r\psi_1-2K_2\cos s\psi_2}, \tag{7c}$$

where

$$\varphi_1 = p_1\psi_1 = \pi/m_1, \quad \varphi_2 = p_2\psi_2 = \pi/m_2 \tag{8}$$

and $W^*_{(j,k)}$ indicates the influence of disorder on $W_{(j,k)}$.

By using the double U-transformation once, the natural vibration equations (1a) and (1b) having $2m_1p_1 \times 2m_2p_2$ unknowns become Eqs. (7a) and (7b) with $2m_1 \times 2m_2$ unknowns (i.e., the displacements of M'-subsystems).

4. The second application of the double U-transformation

It is obvious that the harmonic influence coefficients for the cyclic periodic system possess cyclic periodicity. Consequently, the left-hand sides of the simultaneous equations (7a) possess cyclic periodicity. In order to uncouple the left-hand sides of Eq. (7a), one can now apply again the double U-transformation to Eq. (7a).

Let

$$Q_{(r,s)} = \frac{1}{\sqrt{2m_1}\sqrt{2m_2}}\sum_{j=1}^{2m_1}\sum_{k=1}^{2m_2} e^{-i(j-1)r\varphi_1}e^{-i(k-1)s\varphi_2}W_{(j,k)}, \quad r=1,2,\ldots,2m_1, \quad s=1,2,\ldots,2m_2 \tag{9}$$

and its inverse is

$$W_{(j,k)} = \frac{1}{\sqrt{2m_1}\sqrt{2m_2}}\sum_{r=1}^{2m_1}\sum_{s=1}^{2m_2} e^{i(j-1)r\varphi_1}e^{i(k-1)s\varphi_2}Q_{(r,s)}, \quad j=1,2,\ldots,2m_1, \quad k=1,2,\ldots,2m_2 \tag{10}$$

in which the definitions of φ_1 and φ_2 are the same as those shown in Eq. (8). φ_1, φ_2 denote the periods of the cyclic periodic system for M'-subsystems in x- and y-directions, respectively.

Introducing the double U-transformation (9) into Eqs. (7a)–(7b) and noting the cyclic periodicity of $\beta_{(j,k)(u,v)}$ results in

$$Q_{(r,s)} - (\Delta M\omega^2 - \Delta K)\sum_{u=1}^{2m_1}\sum_{v=1}^{2m_2} e^{-i(u-1)r\varphi_1}e^{-i(v-1)s\varphi_2}\beta_{(u,v)(1,1)}Q_{(r,s)} = Q^*_{(r,s)}, \quad r=1,2,\ldots,2m_1, \quad s=1,2,\ldots,2m_2, \tag{11a}$$

where

$$Q^*_{(r,s)} = \frac{(M_d\omega^2 - K_d)W_{(m_1,m_2)}}{\sqrt{2m_1}\sqrt{2m_2}}\sum_{j=1}^{2m_1}\sum_{k=1}^{2m_2} e^{-i(j-1)r\varphi_1}e^{-i(k-1)s\varphi_2}\beta_{(j,k)(m_1,m_2)}. \tag{11b}$$

Substituting Eq. (7c) into Eqs. (11a)–(11b) yields

$$Q_{(r,s)} - (\Delta M\omega^2 - \Delta K)A(r\varphi_1, s\varphi_2, \omega^2)Q_{(r,s)} = Q^*_{(r,s)}, \quad r=1,2,\ldots,2m_1, \quad s=1,2,\ldots,2m_2 \tag{12a}$$

and

$$Q^*_{(r,s)} = \frac{(M_d\omega^2 - K_d)W_{(m_1,m_2)}}{\sqrt{2m_1}\sqrt{2m_2}} e^{i(1-m_1)r\varphi_1} e^{i(1-m_2)s\varphi_2} A(r\varphi_1, s\varphi_2, \omega^2), \tag{12b}$$

where

$$A(r\varphi_1, s\varphi_2, \omega^2) = \frac{1}{p_1 p_2}\sum_{u=1}^{p_1}\sum_{v=1}^{p_2}\left\{K + 2K_1 + 2K_2 - M\omega^2 - 2K_1\cos[2m_1(u-1)+r]\frac{\varphi_1}{p_1} - 2K_2\cos[2m_2(v-1)+s]\frac{\varphi_2}{p_2}\right\}^{-1}. \tag{12c}$$

For the perfectly cyclic bi-periodic system without any disorder (i.e., $Q^*_{(r,s)} = 0$), the frequency equation can be obtained from Eq. (12a) as

$$1 - (\Delta M\omega^2 - \Delta K)A(r\varphi_1, s\varphi_2, \omega^2) = 0, \quad r = 1, 2, \ldots, 2m_1, \quad s = 1, 2, \ldots, 2m_2. \tag{13}$$

Inserting $Q_{(r,s)}$ obtained from Eqs. (12a) and (12b) into Eq. (10) results in

$$W_{(j,k)} = (M_d\omega^2 - K_d)W_{(m_1,m_2)}\beta^*_{(j,k)(m_1,m_2)}, \quad j = 1, 2, \ldots, 2m_1, \quad k = 1, 2, \ldots, 2m_2 \tag{14}$$

and

$$\beta^*_{(j,k)(m_1,m_2)} = \frac{1}{4m_1 m_2}\sum_{r=1}^{2m_1}\sum_{s=1}^{2m_2}\cos(j-m_1)r\varphi_1\cos(k-m_2)s\varphi_2\frac{A(r\varphi_1, s\varphi_2, \omega^2)}{1 - (\Delta M\omega^2 - \Delta K)A(r\varphi_1, s\varphi_2, \omega^2)}, \tag{15}$$

where $\beta^*_{(j,k)(m_1,m_2)}$ denotes the harmonic influence coefficient for the cyclic bi-periodic system without disorder and means the amplitude of $(j,k)M'$-subsystem caused by unit harmonic force acting at $(m_1, m_2)M'$-subsystem. $W_{(m_1,m_2)}$ is the amplitude of displacement for the disordered subsystem.

5. Frequency equation and localized modes

The frequency equation for the disordered system can be found from Eqs. (14) and (15) with $j = m_1$ and $k = m_2$, as

$$M_d\omega^2 - K_d = \frac{1}{\beta^*_{(m_1,m_2)(m_1,m_2)}} \tag{16a}$$

and

$$\beta^*_{(m_1,m_2)(m_1,m_2)} = \frac{1}{4m_1 m_2}\sum_{r=1}^{2m_1}\sum_{s=1}^{2m_2}\frac{A(r\varphi_1, s\varphi_2, \omega^2)}{1 - (\Delta M\omega^2 - \Delta K)A(r\varphi_1, s\varphi_2, \omega^2)}. \tag{16b}$$

The above frequency equation is applicable to the localized modes of the finite cyclic bi-periodic system with a single disorder. We can now consider the infinite case, namely m_1 and m_2 approach to infinity. The limit of the series summation on the right-hand side of Eq. (15) becomes the double integral and Eq. (15) can be rewritten as

$$\beta^*_{(m_1\pm j,m_2\pm k)(m_1,m_2)} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \cos j\theta_1 \cos k\theta_2 \frac{A(\theta_1,\theta_2,\omega^2)}{1-(\Delta M\omega^2-\Delta K)A(\theta_1,\theta_2,\omega^2)}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2,$$
$$j=0,1,2,\ldots,\infty, k=0,1,2,\ldots,\infty \tag{17a}$$

in which

$$A(\theta_1,\theta_2,\omega^2) \equiv \frac{1}{p_1p_2}\sum_{u=1}^{p_1}\sum_{v=1}^{p_2}\left\{K+2K_1+2K_2-M\omega^2-2K_1\cos\left[\frac{(u-1)2\pi+\theta_1}{p_1}\right]\right.$$
$$\left.-2K_2\cos\left[\frac{(v-1)2\pi+\theta_2}{p_2}\right]\right\}^{-1}. \tag{17b}$$

By introducing Eq. (17a) into Eq. (16a), the frequency equation for the original infinite bi-periodic system shown in Fig. 1 can be found as

$$M_d\omega^2-K_d = \left\{\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{A(\theta_1,\theta_2,\omega^2)}{1-(\Delta M\omega^2-\Delta K)A(\theta_1,\theta_2,\omega^2)}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\right\}^{-1}. \tag{18}$$

When m_1, m_2 approach infinity, the frequency equation (13) for the ordered system becomes the pass band equation as

$$1-(\Delta M\omega^2-\Delta K)A(\theta_1,\theta_2,\omega^2)=0, \quad 0\leqslant\theta_1,\theta_2\leqslant 2\pi. \tag{19}$$

It is obvious that if and only if ω lies in the stop band, namely ω is not any root of Eq. (19), the double integral in Eq. (18) is in existence and the frequency equation (18) is applicable to the localized modes.

Two parameters θ_1 and θ_2 in Eq. (19) indicate two mode phase differences between the two adjacent M'-subsystems in x- and y-directions, respectively. It can be proved that if θ_1, θ_2 are replaced by $2\pi-\theta_1$, $2\pi-\theta_2$, respectively, the function $A(\theta_1,\theta_2,\omega^2)$ shown in Eq. (17b) does not change, leading to the frequency band equation (19) unchanged. Its physical meaning is that four modes with two phase differences θ_1 or $2\pi-\theta_1$, and θ_2 or $2\pi-\theta_2$ correspond to a same frequency. Hence, we consider only the case of $0\leqslant\theta_1,\theta_2\leqslant\pi$ in Eq. (19) and Eqs. (18), (17a) can be rewritten as

$$M_d\omega^2-K_d = \left\{\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\frac{A(\theta_1,\theta_2,\omega^2)}{1-(\Delta M\omega^2-\Delta K)A(\theta_1,\theta_2,\omega^2)}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\right\}^{-1}, \tag{20}$$

$$\beta^*_{(m_1\pm j,m_2\pm k)(m_1,m_2)} = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} \cos j\theta_1 \cos k\theta_2 \frac{A(\theta_1,\theta_2,\omega^2)}{1-(\Delta M\omega^2-\Delta K)A(\theta_1,\theta_2,\omega^2)}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2. \tag{21}$$

When the frequency of localized mode is found from Eq. (20), the corresponding mode can be obtained from Eqs. (14) and (16a), as

$$W_{(m_1\pm j,m_2\pm k)} = W_{(m_1,m_2)}\frac{\beta^*_{(m_1\pm j,m_2\pm k)(m_1,m_2)}}{\beta^*_{(m_1,m_2)(m_1,m_2)}}, \quad j=0,1,2,\ldots,\infty, \quad k=0,1,2,\ldots,\infty \tag{22}$$

in which $W_{(m_1,m_2)}$ acts as an arbitrary constant factor. Applying the well-known Riemann lemma to Eq. (21) yields

$$\lim_{j \text{ or/and } k\to\infty} \beta^*_{(m_1\pm j,m_2\pm k)(m_1,m_2)} = 0, \tag{23}$$

which indicates the mode shown in Eq. (22) is localized.

6. Examples

In order to check the exactness of the formulas and demonstrate how to apply the formulas obtained in the above, some example systems are given as follows.

6.1. Two-dimensional single-periodic system

By letting $p_1 = 1$ and $p_2 = 1$, the original bi-periodic system shown in Fig. 1 becomes single-periodic one, namely all of subsystems are M'-subsystems besides one disorder.

Introducing $p_1 = 1$ and $p_2 = 1$ into Eq. (17b) yields

$$A(\theta_1, \theta_2, \omega^2) = (K + 2K_1 + 2K_2 - M\omega^2 - 2K_1\cos\theta_1 - 2K_2\cos\theta_2)^{-1}. \tag{24}$$

Substituting Eq. (24) into Eqs. (20) and (21) results in

$$M_d\omega^2 - K_d = \left\{\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{1}{K' + 2K_1 + 2K_2 - M'\omega^2 - 2K_1\cos\theta_1 - 2K_2\cos\theta_2}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\right\}^{-1} \tag{25}$$

and

$$\beta^*_{(m_1\pm j,m_2\pm k)(m_1,m_2)} = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{\cos j\theta_1\cos k\theta_2}{K' + 2K_1 + 2K_2 - M'\omega^2 - 2K_1\cos\theta_1 - 2K_2\cos\theta_2}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \tag{26}$$

in which $K' = K + \Delta K$ and $M' = M + \Delta M$.

The localized mode is also shown in Eq. (22) where $\beta^*_{(m_1\pm j,m_2\pm k)(m_1,m_2)}$ should be Eq. (26) instead of Eq. (21).

The frequency equation (25) and the corresponding mode shown in Eqs. (22) and (26) are the same as those given in Ref. [18].

6.2. One-dimensional bi-periodic system

When two arbitrary adjacent subsystems in y-direction are uncoupled, i.e., $K_2 = 0$, the original system is uncoupled into a one-dimensional system having a disorder and many other one-dimensional systems without disorder. We consider now the one-dimensional disordered system.

Substituting $K_2 = 0$ into Eq. (17b) results in

$$A(\theta_1, \theta_2, \omega^2) = A(\theta_1, \omega^2) = \frac{1}{p_1}\sum_{u=1}^{p_1}\left\{K + 2K_1 - M\omega^2 - 2K_1\cos\left[\frac{(u-1)2\pi + \theta_1}{p_1}\right]\right\}^{-1}. \tag{27}$$

Noting that the function $A(\theta_1, \omega^2)$ is independent of θ_2, the frequency equation (20) becomes

$$M_d\omega^2 - K_d = \left\{ \frac{1}{\pi} \int_0^\pi \frac{A(\theta_1, \omega^2)}{1 - (\Delta M\omega^2 - \Delta K)A(\theta_1, \omega^2)} \, \mathrm{d}\theta_1 \right\}^{-1} \tag{28}$$

and the harmonic influence coefficients shown in Eq. (21) can be expressed as

$$\beta^*_{(m_1 \pm j, m_2)(m_1, m_2)} = \frac{1}{\pi} \int_0^\pi \frac{\cos j\theta_1 A(\theta_1, \omega^2)}{1 - (\Delta M\omega^2 - \Delta K)A(\theta_1, \omega^2)} \, \mathrm{d}\theta_1, \quad j = 0, 1, 2, \ldots, \infty, \tag{29a}$$

$$\beta^*_{(m_1 \pm j, m_2 \pm k)(m_1, m_2)} = 0, \quad j = 0, 1, 2, \ldots, \infty, \quad k = 1, 2, \ldots, \infty. \tag{29b}$$

Eq. (22) representing the localized modes becomes

$$W_{(m_1 \pm j, m_2)} = W_{(m_1, m_2)} \frac{\beta^*_{(m_1 \pm j, m_2)(m_1, m_2)}}{\beta^*_{(m_1, m_2)(m_1, m_2)}}, \quad j = 0, 1, 2, \ldots, \infty \tag{30a}$$

and

$$W_{(m_1 \pm j, m_2 \pm k)} = 0, \quad j = 0, 1, 2, \ldots, \infty, \quad k = 1, 2, \ldots, \infty. \tag{30b}$$

These results shown in Eqs. (27), (28), (29a) and (30a) are in agreement with those given in Ref. [21].

6.3. Two-dimensional bi-periodic system with $p_1 = p_2 = 2$

Inserting $p_1 = 2$ and $p_2 = 2$ into Eq. (17b) yields

$$A(\theta_1, \theta_2, \omega^2) = \frac{1}{4} \sum_{u=1}^{2} \sum_{v=1}^{2} \left\{ K + 2K_1 + 2K_2 - M\omega^2 - 2K_1 \cos\left[(u-1)\pi + \frac{\theta_1}{2}\right] - 2K_2 \cos\left[(v-1)\pi + \frac{\theta_2}{2}\right] \right\}^{-1}. \tag{31}$$

Introducing the non-dimensional parameters

$$\Omega^2 \equiv \frac{M\omega^2}{K}, \quad k_1 \equiv \frac{K_1}{K}, \quad k_2 \equiv \frac{K_2}{K} \tag{32}$$

into Eq. (31) results in

$$A(\theta_1, \theta_2, \omega^2) \equiv \frac{1}{K} A_0(\theta_1, \theta_2, \Omega^2), \tag{33a}$$

where

$$A_0(\theta_1, \theta_2, \Omega^2) \equiv \frac{1}{4} \sum_{u=1}^{2} \sum_{v=1}^{2} \left\{ 1 + 2k_1 + 2k_2 - \Omega^2 - 2k_1 \cos\left[(u-1)\pi + \frac{\theta_1}{2}\right] - 2k_2 \cos\left[(v-1)\pi + \frac{\theta_2}{2}\right] \right\}^{-1}. \tag{33b}$$

Substituting Eq. (33a) into Eq. (19), the frequency band equation for the ordered system with $p_1 = 2$ and $p_2 = 2$ can be expressed as

$$1 - (\Delta m\Omega^2 - \Delta k)A_0(\theta_1, \theta_2, \Omega^2) = 0, \quad 0 \leqslant \theta_1, \theta_2 \leqslant \pi, \tag{34a}$$

where

$$\Delta m = \frac{\Delta M}{M}, \quad \Delta k = \frac{\Delta K}{K}. \tag{34b}$$

If the non-dimensional parameters k_1, k_2, Δm and Δk are given, the pass bands can be determined by using a numerical method with a little calculation. For the specific case of

$$k_1 = 0.1, \quad k_2 = 0.2, \quad \Delta m = 0.1, \quad \Delta k = 1 \tag{35}$$

there are two pass bands as follows:

$$[1.09857,\ 2.00000], \quad [2.36364,\ 2.58030] \quad \text{for } \Omega^2 \tag{36a}$$

namely the stop bands are

$$0 < \Omega^2 < 1.09857, \quad 2.00000 < \Omega^2 < 2.36364, \quad 2.58030 < \Omega^2 < \infty. \tag{36b}$$

By introducing Eqs. (33a), (34b) into Eq. (20), the non-dimensional frequency equation for localized vibration can be expressed as

$$F(\Omega^2) = D(\Omega^2), \quad \Omega^2 \in \text{stop bands shown in Eq. (36b)} \tag{37a}$$

where

$$F(\Omega^2) \equiv \varepsilon_M \Omega^2 - \varepsilon_K, \tag{37b}$$

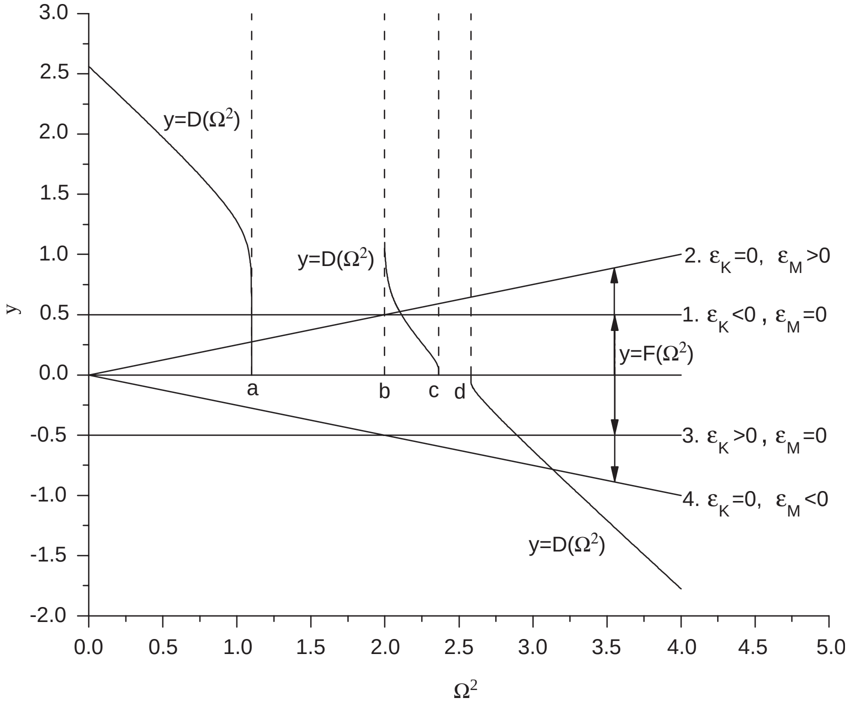
$$D(\Omega^2) \equiv \left\{ \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{A_0(\theta_1, \theta_2, \Omega^2)}{1 - (\Delta m\Omega^2 - \Delta k)A_0(\theta_1, \theta_2, \Omega^2)} \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \right\}^{-1} \tag{37c}$$

and

$$\varepsilon_M \equiv \frac{M_d}{M}, \quad \varepsilon_K \equiv \frac{K_d}{K}. \tag{37d}$$

When the disordered parameters ε_M and ε_K are given besides k_1, k_2, Δm, Δk shown in Eq. (35), the functions, $y = D(\Omega^2)$ and $F(\Omega^2)$, can be plotted against Ω^2. The number of the points of intersection between the two curves is equal to the number of localized modes and the transverse coordinates of the intersection points represent the magnitudes of the non-dimensional frequencies Ω^2.

By using the numerical integral method and the parameters shown in Eq. (35), the curve, D versus Ω^2, is plotted in Fig. 3, which is made up of three continuous and monotonically decreasing curves, corresponding to the three stop bands shown in Eq. (36b). The straight lines, $y = F(\Omega^2)$, corresponding to various disordered parameters, are also plotted in Fig. 3. By observation from Fig. 3, it is clear that in either case of $\varepsilon_K < 0$, $\varepsilon_M = 0$ or $\varepsilon_K = 0$, $\varepsilon_M > 0$, two localized modes will occur and the corresponding frequencies lie in the first and second stop bands and in either case of $\varepsilon_K > 0$, $\varepsilon_M = 0$ or $\varepsilon_K = 0$, $\varepsilon_M < 0$, only one localized mode with higher frequency in the third stop band will occur.



$a = 1.09857$, $b = 2.00000$, $c = 2.36364$, $d = 2.58030$

Fig. 3. Functions $D(\Omega^2)$ and $F(\Omega^2)$ for $p_1 = p_2 = 2$.

Consider now the case of $K_d = 0.5K$, $M_d = 0$, i.e.,

$$\varepsilon_K = 0.5, \quad \varepsilon_M = 0. \tag{38}$$

By means of the numerical integral method and graphic representation, the solution of the frequency equation (37a)–(37d) for Ω^2 can be found as

$$\Omega^2 = 2.89128. \tag{39}$$

Recalling Eqs. (21), (22), (33a), the localized mode can be expressed as

$$W_{(m_1+j,m_2+k)} = \frac{I_{(j,k)}(\Omega^2)}{I_{(0,0)}(\Omega^2)}, \quad j,k = 0, \pm 1, \pm 2, \ldots, \infty, \tag{40a}$$

where

$$I_{(j,k)} \equiv K\beta^*_{(m_1+j,m_2+k)(m_1,m_2)} = \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \frac{\cos j\theta_1 \cos k\theta_2 A_0(\theta_1,\theta_2,\Omega^2)}{1-(\Delta m\Omega^2 - \Delta k)A_0(\theta_1,\theta_2,\Omega^2)}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2. \tag{40b}$$

The localized mode shown in Eqs. (40a)–(40b) has been normalized according to the condition of $W_{(m_1,m_2)} = 1$. Substituting Eq. (39) into Eqs. (40a)–(40b) and using the numerical integral method, the localized mode can be found as shown in Table 1. The numerical results show that the amplitude ratios between the two adjacent M'-subsystems in x- and y-directions are not constants,

Table 1
Localized mode with $\Omega^2 = 2.89128$

(a) $W_{(m_1 \pm j, m_2)}$							
j	0	1	2	3	4	5	6
W	1	0.01958	0.0004261	9.850E − 06	2.373E − 07	5.891E − 09	1.495E − 10
(b) $W_{(m_1, m_2 \pm k)}$							
k	0	1	2	3	4	5	6
W	1	0.06587	0.004447	0.0003040	2.096E − 05	1.455E − 06	1.016E − 07

namely $W_{(m_1+j+1,m_2)}/W_{(m_1+j,m_2)}$ $(j = 0, 1, 2, \ldots)$ and $W_{(m_1,m_2+k+1)}/W_{(m_1,m_2+k)}$ $(k = 0, 1, 2, \ldots)$ are not constants. This property is different from that for one-dimensional periodic systems.

7. Conclusions

In this work, the application of the U-transformation method has been extended to the analysis of localized modes from one-dimensional bi-periodic mass–spring systems to two-dimensional systems. In order to utilize completely the property of bi-periodicity in two-dimensional systems, the proposed method requires the application of the double U-transformation twice. The governing equation of natural vibration is uncoupled to form a set of single degree of freedom equations in terms of the harmonic influence coefficients. As a result the frequency equation of the disordered system and localized modes can be derived.

Two special cases, two-dimensional single-periodic and one-dimensional bi-periodic systems, are considered. The results for two cases are in agreement with those given in the literature. A specific two-dimensional bi-periodic system with $p_1 = p_2 = 2$ is taken as example. The amplitude ratios of localized modes between the two adjacent M'-subsystems are not only different in x- and y-directions but also variable along x- and y-directions. However, it is well known that the corresponding amplitude ratio in one-dimensional periodic system is a constant. This is the great difference between one- and two-dimensional periodic systems.

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